Availability of a Multi-Unit Repairable System

Jyotirmoy Sarkar

Indiana University Purdue University
Indianapolis

February, 2006

Collaborators

Bruno Beith **Liang Hong**

Outline of Talk

- One-Unit System
- Two-Unit Series System
- n-Unit Series System
- Various Maintenance Models under exponential life and repair times
- Two-Unit Parallel System
- $2|3$ (F) System
- Future Research

One-Unit System

State-Time Diagram

Assume

(1) instantaneous commencement of repair

(2) instantaneous installation of repaired unit

(3) perfect repair, and

Lifetimes: $X_1, X_2, \ldots \overset{IID}{\sim} F$

Repair times: $Y_1, Y_2, \ldots \overset{IID}{\sim} G$

Define

Instantaneous availability

$$A(t) = P\{\text{system is up at time } t\}$$

Steady-state availability

$$A = \lim_{t \to \infty} A(t)$$

One-Unit System (contd.)

Let $X + Y \sim F * G \equiv H$. Then

$$A(t) = \bar{F}(t) + \int_0^t \bar{F}(t-u)\, d\left(\sum_{n=1}^{\infty} H^{(n)}\right)(u)$$

$$\boxed{\boxed{\begin{array}{c}\text{Key Renewal Theorem (Smith, 1958)} \Rightarrow \\ \lim_{t\to\infty} A(t) = \dfrac{\int_0^\infty \bar{F}(x)\, dx}{\int_0^\infty \bar{H}(u)\, du} = \dfrac{\mu_F}{\mu_F + \mu_G}\end{array}}}$$

Hence,

$$A = \frac{E[X]}{E[X] + E[Y]} = \frac{\text{MTTF}}{\text{MTTF+MTTR}}$$

$\boxed{\text{QUIZ}}$ Keep rolling a fair die. Find P[CUSUM=10^6]

For example, $5 + 4 + 1 + 6 + 6 + 5 + \ldots$
has CUSUM 5, 9, 10, 16, 22, 27, ...

One-Unit System (contd.)

$A(t)$ is typically much harder. One easy case is

(1) $F \equiv$Exp(λ) and $G \equiv$Exp(μ)

$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu}\, e^{-(\lambda+\mu)\, t}$$

(1975) Barlow & Proschan
(1994) Høyland & Rausand

(2) $F \equiv$Gamma($2, \lambda$) and $G \equiv$Exp(μ)
Letting

$$l = \lambda + \frac{\mu}{2} \quad \text{and} \quad k = \frac{1}{2}\sqrt{\mu(4\lambda - \mu)}$$

$$A(t) = \frac{2\mu}{\lambda + 2\mu} + \frac{\lambda}{\lambda + 2\mu}\, e^{-l\, t}\, \left[\cos(kt) + \frac{l}{k}\sin(kt)\right]$$

(1999) Sarkar & Chaudhuri

$\boxed{A(t) \text{ via Modern Technique}}$

Define Fourier Transform

$$\boxed{\tilde{f}(s) = \int_{-\infty}^{\infty} e^{i\,s\,t}\, f(t)\, dt}$$

THEOREM 1

Let $B(t) \equiv 1 - A(t)$ and $b(t) \equiv B'(t)$. Then

$$\begin{aligned} B(t) &\equiv P[\text{unit down at time } t] \\ &= P[X_1 \le t < X_1 + Y_1] + \\ &\qquad P[X_1 + Y_1 \le t, \text{unit down at time } t] \\ &= F(t) - H(t) + \int_0^t B(t-v)\, dH(v) \end{aligned}$$

$\Rightarrow$ (derivative)

$$b(t) = f(t) - h(t) + \int_0^t b(t-v)\, h(v)\, dv$$

$\Rightarrow$ (Fourier transform)

$$\tilde{b}(s) = \tilde{f}(s) - \tilde{h}(s) + \tilde{b}(s)\, \tilde{h}(s)$$

or,

$$\tilde{b}(s) = \frac{\tilde{f}(s) - \tilde{h}(s)}{1 - \tilde{h}(s)} = \tilde{f}(s)\, \frac{1 - \tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}$$

since $\tilde{h}(s) = \tilde{f}(s)\tilde{g}(s)$

$\Rightarrow$ (Inversion Formula, e.g. Kaplan, 1984)

$$b(t) \;=\; (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\,t\,s}\, \tilde{b}(s)\, ds$$

If $c_t(z)$ has singularities in the LHP, then

$$b(t) \;=\; (2\pi)^{-1} \int_{-\infty}^{\infty} c_t(s)\, ds$$

$$=\; -i \sum_{z_j \in \mathrm{LHP}} \mathrm{Res}_{z_j}\; c_t(z)$$

where $\mathrm{Res}_{z_j}\; c_t(z) = \ldots$

Finally,

$$A(t) \;=\; 1 - \int_0^t b(u)\, du$$

Two-Unit Series System

Diagram

$$A_S(t) = P\{\text{system is up at time } t\}$$

Model 1: Units behave independently

$$A_{S1}(t) = A_1(t)\ A_2(t)$$

very like the reliability formula $R_S = R_1\ R_2$

Units behave independently means
(1) two units cannot fail together
(2) when one fails, the other continues on
(3) two repair facilities needed
(4) unit installed as soon as repaired

In practice, maintenance policy is different, rendering the product rule inappropriate.

Model F: Nonfailed unit frozen

(1) two units cannot fail together
(2) when one fails, the other is frozen
(3) one repair facility suffices
(4) unit installed as soon as repaired
Sherwin (2000) ⇒

$$\begin{aligned}
A_{SF}(t) &= \frac{P\{\text{system up at time } t\}}{P\{\text{all possible states at time } t\}} \\
&= \frac{P\{\text{unit 1 } \textbf{and} \text{ unit 2 up at } t\}}{P\{\text{unit 1 } \textbf{or} \text{ unit 2 up at } t\}} \\
&= \frac{A_1(t)\ A_2(t)}{A_1(t) + A_2(t) - A_1(t)\ A_2(t)} \\
&= \frac{A_1(t)\ A_2(t)}{A_1(t)\ A_2(t) + \bar{A}_1(t))\ A_2(t) + A_1(t)\ \bar{A}_2(t))} \\
&= \left[1 + \frac{\bar{A}_1(t)}{A_1(t)} + \frac{\bar{A}_2(t)}{A_2(t)}\right]^{-1}
\end{aligned}$$

where $\bar{A}_i(t) = 1 - A_i(t)$

n-Unit Series System

Diagram

Model 1: Units behave independently

$$A_{S1}(t) = \prod_{i=1}^{n} A_i(t)$$

very like the reliability formula $R_S = \Pi_{i=1}^{n} R_i$.

When one unit fails, the other continues on, and $r = n$ repair facilities needed.

In practice, when one unit fails, the others are frozen, so $r = 1$ repair facility suffices

Model F: Nonfailed units frozen

$$A_{SF}(t) = \left[1 + \sum_{i=1}^{n} \frac{\bar{A}_i(t)}{A_i(t)}\right]^{-1}$$

Commentaries

- David Sherwin (2000) IEEE TR 49(2):

“Some (not all) books, and some expensive courses on Reliability, suggest that the steady-state availability of a series system is found by the product rule as for reliability. This is wrong.”

“My objective in this paper is to stimulate debate so that the assumptions and restrictions often made, will emerge, and some standard text-books will be challenged.”

- Hoang Pham (2003) IEEE TR 52(2):

“The result shows that there are important consequences on steady-state availability computations based on how one models the aging characteristics of nonfailed components during a failure.”

Two-Unit Series System, under exponential lifetimes and repair times

Assume, for unit $i = 1, 2$

$$\text{Lifetimes} \overset{IID}{\sim} \text{Exponential}(\lambda_i)$$
$$\text{Repair times} \overset{IID}{\sim} \text{Exponential}(\mu_i)$$

In this case

$$A_i = \frac{\mu_i}{\lambda_i + \mu_i}$$

At any time, each unit is	
1	up and operating
1_s	up and on stand by
0	down and being repaired
0_w	down and waiting for repair

State of the system δ
For example, $\delta = (0, 0_w)$ means ...
The system is up only in state $\delta_1 = (1, 1)$

$\boxed{\text{Markov Chain}}$

has finite state space $\Omega = \{\delta_1, \delta_2, \ldots, \delta_M\}$

Let $P_{\delta_j \delta_k} = P\{\text{state } (t) = \delta_j | \text{state } (0) = \delta_k\}$

Let $\mathbf{P}(t) = \Big(\Big(P_{\delta_j \delta_k}(t)\Big)\Big)$ Then $\mathbf{1}^T \, \mathbf{P}(t) = \mathbf{1}^T$

Let $\mathbf{Q} = ((q_{\delta_j \delta_k}))$ be the infinitesimal transition rate matrix; that is, transition time from state δ_j to state δ_k is exponential $(q_{\delta_j \delta_k})$, and $q_{\delta_j \delta_j} = - \Sigma_{k \neq j} \, q_{\delta_j \delta_k}$

Note that $\mathbf{Q} \; \mathbf{1} = \mathbf{0}$

$\boxed{\text{Kolmogorov's Forward Equations}}$

$$\mathbf{Q}^T \, \mathbf{P}(t) = \frac{d}{dt} \mathbf{P}(t)$$

... (via Fourier transformation and inversion)

$$\mathbf{P}(t) = \sum_{k=0}^{\infty} (\mathbf{Q}^T)^k \, \frac{t^k}{k!}$$

Cox and Miller (1965)

In particular, taking $t \longrightarrow \infty$, we obtain the steady-state probabilities $\mathbf{P}(\infty) = [\mathbf{\Pi}| \ldots |\mathbf{\Pi}]$ by solving

$$\mathbf{1}^T \, \mathbf{\Pi} = 1 \quad \text{and} \quad \mathbf{Q}^T \, \mathbf{\Pi} = \mathbf{0}$$

Replace top row of $\mathbf{Q}^T$ by $\mathbf{1}^T$ to get $\mathbf{R}$,
and partition $\mathbf{R}$ as

$$\begin{bmatrix} 1 & 1 \ldots 1 \\ \mathbf{r}_{21} & \mathbf{R}_{22} \end{bmatrix}$$

Now solve for $\mathbf{\Pi}$ in

$$\mathbf{R} \, \mathbf{\Pi} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Recall $\delta_1 = (1, 1)$ is the only up state. So

$$A_S = \mathbf{\Pi}_{\delta_1} = r^{11} = \left\{1 - \mathbf{1}^T \, \mathbf{R}_{22}^{-1} \, \mathbf{r}_{21}\right\}^{-1}$$

In various maintenance models, we compute

$$A_S^{-1} = 1 + \mathbf{1}^T \, (-\mathbf{R}_{22})^{-1} \, \mathbf{r}_{21}$$

$\boxed{\text{Model F: (F, } r = 1\text{, IASAR)}}$

$\Omega_{SF} = \{11, 01, 10\}$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 \\ \mu_1 & -\mu_1 & 0 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & -\mu_1 & 0 \\ \lambda_2 & 0 & -\mu_2 \end{bmatrix}$$

$$\begin{aligned} A_{SF}^{-1} &= 1 + (1 \;\; 1) \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} \end{aligned}$$

agrees with expression earlier (page 9).

Sandler (1963) gave A_{SF} for n-unit series system, under exponential lifetimes and repair times.

$$\boxed{\text{Model 1: (DNF, } r = 2\text{, IASAR)}}$$

$$\Omega_1 = \{11, 01, 10, 00\}$$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1 + \lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\lambda_1 + \mu_2) & \lambda_1 \\ 0 & \mu_2 & \mu_1 & -(\mu_1 + \mu_2) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & -(\mu_1 + \lambda_2) & 0 & \mu_2 \\ \lambda_2 & 0 & -(\lambda_1 + \mu_2) & \mu_1 \\ 0 & \lambda_2 & \lambda_1 & -(\mu_1 + \mu_2) \end{bmatrix}$$

$$
\begin{aligned}
A_{S1}^{-1} &= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1 + \lambda_2 & 0 & -\mu_2 \\ 0 & \lambda_1 + \mu_2 & -\mu_1 \\ -\lambda_2 & -\lambda_1 & \mu_1 + \mu_2 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \\
&= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} (\lambda_1 + \mu_1 + \mu_2)\mu_2 & \lambda_1\mu_2 & * \\ \lambda_2\mu_1 & \mu_1(\mu_1 + \mu_2 + \lambda_2) & * \\ \lambda_2(\lambda_1 + \mu_2) & \lambda_1(\mu_1 + \lambda_2) & * \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \; / \; D_1
\end{aligned}
$$

where $D_1 = (\mu_1 + \lambda_2)(\lambda_1 + \mu_1 + \mu_2)\mu_2 - \mu_2\lambda_2(\lambda_1 + \mu_2)$

$$
\begin{aligned}
&= 1 + \frac{\lambda_1\mu_2 + \lambda_2\mu_1 + \lambda_1\lambda_2}{\mu_1\mu_2} \\
&= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2}
\end{aligned}
$$

agrees with expression earlier (page 8), since

$$
A_{S1}^{-1} = A_1^{-1} \, A_2^{-1} = \left[1 + \frac{\lambda_1}{\mu_1}\right] \; \left[1 + \frac{\lambda_2}{\mu_2}\right]
$$

$$\boxed{\text{Model 2: (DNF, } r=1\text{, IASAR, RIOF)}}$$

$$\Omega_{S2} = \{11, 01, 10, 00_w, 0_w0\}$$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & 0 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & \mu_2 & 0 & 0 & -\mu_2 \end{bmatrix}$$

$$\begin{aligned} A_{S2}^{-1} &= 1 + (1 \;\; 1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1+\lambda_2 & 0 & 0 & -\mu_2 \\ 0 & \lambda_1+\mu_2 & -\mu_1 & 0 \\ -\lambda_2 & 0 & \mu_1 & 0 \\ 0 & -\lambda_1 & 0 & \mu_2 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix} \\ &= 1 + (1 \;\; 1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1\mu_2(\lambda_1+\mu_2) & \lambda_1\mu_1\mu_2 & * & * \\ \lambda_2\mu_1\mu_2 & \mu_1\mu_2(\mu_1+\lambda_2) & * & * \\ \lambda_2\mu_2(\lambda_1+\mu_2) & \lambda_1\lambda_2\mu_2 & * & * \\ \lambda_1\lambda_2\mu_1 & \lambda_1\mu_1(\mu_1+\lambda_2) & * & * \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix} \;/\; D_2 \\ & \qquad \text{where } D_2 = \mu_1\mu_2[\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2] \\ &= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2}\left[1 + \frac{\lambda_1\mu_2 + \lambda_2\mu_1 + \mu_1\mu_2}{\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2}\right] \end{aligned}$$

$$\boxed{\text{Model 3: (DNF, } r=1\text{, IASAR, RLLF)}}$$

WLOG assume $\lambda_1^{-1} \geq \lambda_2^{-1}$. Then $\Omega_{S3} = \{11, 01, 10, 00_w\}$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 \end{bmatrix}$$

$$\begin{aligned} A_{S3}^{-1} &= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1+\lambda_2 & 0 & 0 \\ 0 & \lambda_1+\mu_2 & -\mu_1 \\ -\lambda_2 & -\lambda_1 & \mu_1 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \\ &= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1\mu_2 & 0 & * \\ \lambda_2\mu_1 & \mu_1(\mu_1+\lambda_2) & * \\ \lambda_2(\lambda_1+\mu_2) & \lambda_1(\mu_1+\lambda_2) & * \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \;/\; D_3 \\ & \qquad \text{where } D_3 = \mu_1\mu_2(\mu_1+\lambda_2) \\ &= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2}\left[1 + \frac{\mu_1+\lambda_1}{\mu_1+\lambda_2}\right] \end{aligned}$$

$\boxed{\text{Model 4: (DNF, } r = 1\text{, IWBAR)}}$

When both units down, order of repair irrelevant
$\Omega_{S4} = \{11, 01, 10, 00_w, 1_s0\}$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & \lambda_1 & 0 \\ 0 & 0 & 0 & -\mu_1 & \mu_1 \\ \mu_2 & 0 & 0 & 0 & -\mu_2 \end{bmatrix}$$

$$\begin{aligned} A_{S4}^{-1} &= 1 + (1\ \ 1\ \ 1\ \ 1) \begin{bmatrix} \mu_1+\lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1+\mu_2 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & \mu_1 & 0 \\ 0 & 0 & -\mu_1 & \mu_2 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix} \\ &= 1 + (1\ \ 1\ \ 1\ \ 1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda_2/\mu_1 & \lambda_1/\mu_1 & 1/\mu_1 & 0 \\ \lambda_2/\mu_2 & \lambda_1/\mu_2 & 1/\mu_2 & 1/\mu_2 \end{bmatrix} \begin{pmatrix} \frac{\lambda_1}{\mu_1+\lambda_2} \\ \frac{\lambda_2}{\lambda_1+\mu_2} \\ 0 \\ 0 \end{pmatrix} \\ &= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2}\left[\frac{\mu_1}{\mu_1+\lambda_2} + \frac{\mu_2}{\lambda_1+\mu_2}\right] \end{aligned}$$

$$\boxed{\text{Model 5: (DNF, } r = 2\text{, IWBAR)}}$$

$$\Omega_{S5} = \{11, 01, 10, 00_w, 1_s0, 01_s\}$$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & -(\mu_1+\mu_2) & \mu_1 & \mu_2 \\ \mu_2 & 0 & 0 & 0 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 & 0 & 0 & -\mu_1 \end{bmatrix}$$

$$\begin{aligned} A_{S4}^{-1} &= 1 + (1 \;\; 1 \;\; 1 \;\; 1 \;\; 1) \begin{bmatrix} \mu_1+\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1+\mu_2 & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & \mu_1+\mu_2 & 0 & 0 \\ 0 & 0 & -\mu_1 & \mu_2 & 0 \\ 0 & 0 & -\mu_2 & 0 & \mu_1 \end{bmatrix}^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \ldots \\ &= 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1\lambda_2}{\mu_1\mu_2}\left[\frac{\mu_1^2}{\mu_1+\lambda_2} + \frac{\mu_2^2}{\lambda_1+\mu_2}\right] \frac{1}{\mu_1+\mu_2} \end{aligned}$$

Comparing Models for 2-Unit Series System

$$A_{Sj}^{-1} = 1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2}\, \theta_j$$

sorted as below

j	r	IWR	R order	θ_j
DNF				
2	1	1	IOF	$1 + \frac{\lambda_1\mu_2+\lambda_2\mu_1+\mu_1\mu_2}{\lambda_1\mu_1+\lambda_2\mu_2+\mu_1\mu_2}$
3	1	1	LLF	$1 + \frac{\mu_1+\lambda_1}{\mu_1+\lambda_2}$ assuming $\lambda_1<\lambda_2$
4	1	2	irrel	$\frac{\mu_1}{\mu_1+\lambda_2} + \frac{\mu_2}{\lambda_1+\mu_2}$ $>$ iff $\lambda_1\lambda_2 < \mu_1\mu_2$
1	2	1	simult	1
5	2	2	simult	$\left[\frac{\mu_1^2}{\mu_1+\lambda_2} + \frac{\mu_2^2}{\lambda_1+\mu_2}\right] \frac{1}{\mu_1+\mu_2}$
F	1	1	N/A	0

Availability of n-Unit Series System

Assume exponential lifetimes and repair times

M	r	IWR	R order	A_{SM}
Ind	n	1	simult	$\Pi_i \left(1 + \frac{\lambda_i}{\mu_i}\right)$
DNF	$r < n$	all	?	??
Freeze	1	1	N/A	$1 + \Sigma_i \frac{\lambda_i}{\mu_i}$

Can you generalize Model 4 for $n > 2$ units?

Two-Unit Parallel System

Diagram

$$A_{PM}(t) = P\{\text{parallel system is up at time } t\}$$

Model I: Units behave independently

(DNF, $r = 2$, IASAR)

$$\begin{aligned} A_{PI}(t) &= 1 - P\{\text{unit 1 } \textbf{and} \text{ unit 2 are down at time } t\} \\ &= 1 - \bar{A}_1(t)\ \bar{A}_2(t) \end{aligned}$$

$$\begin{aligned} A_{PI} &= 1 - P\{\text{unit 1 } \textbf{and} \text{ unit 2 are down}\} \\ &= 1 - \bar{A}_1\ \bar{A}_2 \end{aligned}$$

very like the reliability formula $R_P = 1 - \bar{R}_1\ \bar{R}_2$
Under exponential lifetime and repair times

$$A_{PI}^{-1} = 1 + \frac{\lambda_1 \lambda_2}{\lambda_1 \mu_2 + \mu_1 \lambda_2 + \mu_1 \mu_2}$$

$\boxed{\text{Model D: (DNF, } r = 1\text{, IASAR)}}$

Assume exponential lifetime and repair times
When both units fail, repair quicker one first

WLOG assume $\mu_1^{-1} \leq \mu_2^{-1}$. Then, as in Model 4, $\Omega_{S3} = \{11, 01, 10, 00_w\}$

$$\mathbf{Q} = \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1 + \lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\lambda_1 + \mu_2) & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 \end{bmatrix}$$

$$\begin{aligned} P_{00_w}^{-1} &= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} \lambda_1 + \lambda_2 & -\mu_1 & \mu_2 \\ -\lambda_1 & \mu_1 + \lambda_2 & 0 \\ -\lambda_2 & 0 & \lambda_1 + \mu_2 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \mu_1 \end{pmatrix} \\ &= 1 + (1 \;\; 1 \;\; 1) \begin{bmatrix} * & * & \mu_2(\mu_1 + \lambda_2) \\ * & * & \lambda_1 \; \mu_2 \\ * & * & \lambda_2(\lambda_1 + \lambda_2 + \mu_1) \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \mu_1 \end{pmatrix} \; / \; D \\ &\qquad \text{where } D = \lambda_1 \; \lambda_2 \; (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) \\ &= 1 + \frac{\mu_1 \; (\lambda_2 + \mu_2) \; (\lambda_1 + \lambda_2 + \mu_1)}{\lambda_1 \; \lambda_2 \; (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)} \end{aligned}$$

$A_{PD} = 1 - P_{00_w}$ implies

$$A_{PD}^{-1} = 1 + \frac{\lambda_1 \; \lambda_2 \; (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)}{\mu_1 \; (\lambda_2 + \mu_2) \; (\lambda_1 + \lambda_2 + \mu_1)} > A_{PI}^{-1}$$

$\boxed{n\text{-Unit Parallel System}}$

Diagram

$\boxed{\text{Model I: Units behave independently}}$

(DNF, $r = n$, IASAR)

$$\begin{aligned} A_{PI}(t) &= 1 - P\{\textbf{all} \text{ units are down at time } t\} \\ &= 1 - \prod_i \bar{A}_i(t) \end{aligned}$$

$A_{PI} = 1 - \Pi_i \bar{A}_i$
very like the reliability formula $R_P = 1 - \Pi_i \bar{R}_i$

What about Model D: (DNF, $r < n$, IASAR)?

$\boxed{2|3(F)\text{ System}}$

Diagram

$\boxed{\text{Model I: Units behave independently}}$

(DNF, $r = 3$, IASAR)

$$\begin{aligned} A_{2|3(F),I}(t) &= A_1(t)A_2(t)A_3(t) + \bar{A}_1(t)A_2(t)A_3(t) \\ &\quad + A_1(t)\bar{A}_2(t)A_3(t) + A_1(t)A_2(t)\bar{A}_3(t) \end{aligned}$$

$$A_{2|3(F),I} = A_1A_2A_3 + \bar{A}_1A_2A_3 + A_1\bar{A}_2A_3 + A_1A_2\bar{A}_3$$

very like the reliability formula

$$R_{2|3(F)} = R_1R_2R_3 + \bar{R}_1R_2R_3 + R_1\bar{R}_2R_3 + R_1R_2\bar{R}_3$$

$\boxed{\ 2|3(F)\text{ System}}$

$\boxed{\text{Model R2: (F2, } r = 2\text{, IASAR)}}$

When two units fail, freeze the third.
$\Omega_{2|3(F),R2} = \{111, 011, 101, 110, 001, 010, 100\}$
First four up states, last three down states

$$A_{2|3(F),R2} = \frac{A_1A_2A_3 + \bar{A}_1A_2A_3 + A_1\bar{A}_2A_3 + A_1A_2\bar{A}_3}{1 - \bar{A}_1\bar{A}_2\bar{A}_3}$$

Furthermore, if exponential life and repair

$$\mathbf{Q} = \begin{bmatrix} * & \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \mu_1 & * & 0 & 0 & \lambda_2 & \lambda_3 & 0 \\ \mu_2 & 0 & * & 0 & \lambda_1 & 0 & \lambda_3 \\ \mu_3 & 0 & 0 & * & 0 & \lambda_1 & \lambda_2 \\ 0 & \mu_2 & \mu_1 & 0 & * & 0 & 0 \\ 0 & \mu_3 & 0 & \mu_1 & 0 & * & 0 \\ 0 & 0 & \mu_3 & \mu_2 & 0 & 0 & * \end{bmatrix}$$

$$A^{-1}_{2|3(F),R2} = 1 + \frac{\lambda_1\lambda_2\mu_3 + \lambda_1\mu_2\lambda_3 + \mu_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3 + \lambda_1\mu_2\mu_3 + \mu_1\lambda_2\mu_3 + \mu_1\mu_2\lambda_3}$$

$2|3(F)$ System

Model R1: (F2, $r = 1$, IASAR)

Order of repair: unit 1 before unit 2 before unit 3
(iff $\mu_1/\lambda_1 < \mu_2/\lambda_2 < \mu_3/\lambda_3$???)
$\Omega_{2|3(F),R1} = \{111, 011, 101, 110, 00_w1, 010_w, 100_w\}$

first four up states, last three down states

If exponential life and repair

$$\mathbf{Q} = \begin{bmatrix} * & \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ \mu_1 & * & 0 & 0 & \lambda_2 & \lambda_3 & 0 \\ \mu_2 & 0 & * & 0 & \lambda_1 & 0 & \lambda_3 \\ \mu_3 & 0 & 0 & * & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \mu_1 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & * & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & * \end{bmatrix}$$

$$A_{2|3(F),R1} = ?$$

Future Research

Study the following systems

- series-parallel
- parallel-series
- $k|n(F)$ with $k < n$
- bridge

Vary r

Allow nonexponential life and repair times

Questions:

When should nonfailed units be frozen?

What's the order of repair when $> r$ units fail?

When to install a repaired unit to operation?

Statistical inference on A?

THANK YOU